THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN,
AND BOMBAY.

VOL. IV. MARCH, 1907. No. 62.

REPORT OF THE M.A. COUNCIL FOR 1906.

DURING the past year 32 new members of the Association have been elected, and 13 have resigned. Only one member, so far as the Council is aware, has been removed by death. Miss Beale, of Cheltenham, had been a member of the Association since 1874, and had always evinced great interest in its work and welfare. The Council desires to record its deep sense of the loss which has been sustained, not only by this Association, but by all who are concerned in the best interests of educational progress.

The Association now consists of 419 members, including 10 honorary members, 24 life members by composition, and 73 who are exempt from payment of the annual subscription under a rule which has now been rescinded.

In March a communication was received from the Teachers' Guild in which the Council was invited to send delegates to a preliminary meeting to be called to consider the desirability of holding a general Educational Congress, annually, in the month of January, for the discussion of purely educational subjects. The Council appointed Professor G. B. Mathews, Mr. C. S. Jackson, and Mr. C. E. Williams to be its representatives, and the preliminary meeting was held on May 17th. The Council of this Association met again on July 11th, and, after receiving and considering its representatives' reports, passed the following resolution: "That the Council of the Mathematical Association, while prepared to consider any definite scheme for the co-operation of Educational Associations, decides to take no action at present."

A letter was received also from Miss C. M. Mason, of the House of Education, Ambleside, on behalf of the Parents' National Educational Union. Miss Mason suggested that the Mathematical Association should appoint a committee to set work—term by

term—in Arithmetic, Algebra and Geometry, and to set and examine a test paper in the work of each subject set in each school. After careful consideration the Council came to the conclusion that the Association could not undertake work of this character.

A request was received from the Rev. E. F. M. Macarthy, on behalf of the Head-masters' Association, asking for a conference with our "Teaching of Elementary Mathematics" Committee, but unfortunately a convenient date was not chosen. The Council hopes that a meeting may be arranged this year.

The Treasurer's statement for 1906 will presently be laid before the meeting, but the Council desires specially to report that two days ago and since the accounts were closed the Association has received from Professor R. B. Clifton, of Oxford, one of our honorary members, a donation of £10. The Council asks that the warmest thanks of the Association be tendered to Professor Clifton for his generous gift.

The second annual dinner of the Association was held at the Criterion Restaurant in the evening, after the general meeting on January 26th, and about 25 members and guests were present. The President (Mr. G. B. Mathews), Mr. E. M. Langley, Mr. R. W. K. Edwards, and Mr. Alan Walker (representative of Messrs. Underwood & Underwood) were invited to the dinner as guests of the Association. The arrangements were admirable, and the Council desires to express its thanks again to Mr. C. S. Jackson for the trouble which he took in organising them. The third annual dinner will take place this evening.

The Council regrets that Mr. Mathews' two years of office have now come to an end, and it desires to offer him the warmest thanks of the Association for the manner in which he has watched over its interests and welfare during this period. As Mr. Mathews' successor the Council nominates Professor G. H. Bryan, F.R.S., of University College, Bangor.

The two retiring members of the Council are Mr. W. N. Roseveare of Harrow, and Mr. C. O. Tuckey of Charterhouse, and the Council has much pleasure in nominating Mr. C. A. Rumsey of Dulwich, and Mr. W. J. Dobbs to succeed them.

Jan. 26, 1907.

MEETING OF THE ASSOCIATION.

THERE was a good discussion on contracted multiplication and division. Mr. Lodge suggested that it would probably be easier to introduce beginners to the advantages of contraction in long division before using it in multiplication, as in the latter case the simplicity of contracting is obscured by the need of commencing to multiply by the high figures of the multiplier, a method which

ought to be taught in preparatory schools, or even earlier, but is not.

He called attention also to the method or rather 'tip' which he had suggested in the January number of the *Gazette*, for avoiding error in the course of the work due to forgetting how many figures of the divisor (or multiplicand) were still to be used, viz., by writing each operating digit, in its turn, immediately over or under the figure on which it begins operations: by this means all strain of remembering how many figures of divisor (or multiplicand) are still active at any stage of the work is absolutely removed. There is no strain, no hesitation, and each contraction is obvious and easy. He recommended the 'tip' to all who had felt contractions in any way fatiguing, not to be insisted on in teaching, but as a handy help when desired

Dr. Macaulay, and some of the other speakers, thought that when contracted work was wanted it was better to use logarithms: Mr. Hawkins, on the other hand, held that contractions were useful in cases where it was not worth while getting out a logarithm book, and said that boys soon came to enjoy contractions so long as they were not taught as a 'method' but as common sense. He thought just scratching out figures which were done with was the best way. Mr. Siddons said it was not worth while teaching contracted methods, as they were soon forgotten unless in constant use: he himself had struggled for three years with them at school, he had never used them at college, and afterwards had to learn them again for teaching. Prof. Barrell thought that very little time was needed in order to learn how to contract, as the method was so obvious provided the boy was already accustomed to multiply by the highest digit. Other speakers besides Mr. Siddons spoke of the difficulty of being always sure that the last figure retained was the *nearest*; and if an answer with the last figure approximately correct was considered sufficiently good the objection to contractions would disappear.

Mr. Jackson spoke warmly in favour of Mr. Lodge's method, and mentioned other men who thought it constituted an important advance. The President spoke in favour of contractions, especially in connection with experimental or other practical work, as the tail figures which came in otherwise were liable to be thought of as important, whereas they were absolutely false and misleading.

Finally, by resolution, the whole matter was referred to the Committee on Mathematical teaching (of which Mr. Siddons is secretary), with a strong recommendation that all possible steps should be taken to urge on Preparatory Schools the need of teaching boys from the beginning to multiply by the highest digit, and also, in subtraction, to use the 'complementary' or 'shop' method.

THE NEGLECTED BRITISH TEACHER.

A PLEA FOR ORGANISATION IN MATHEMATICS.

A Paper sent by the President-Elect to be read at the Annual Meeting, January 26, 1907.

A SHORT time ago I happened to refer to our list of members, and I thought it would be interesting to find out what proportion of the total was contributed by Wales, and how far the teachers in Welsh secondary and other schools had availed themselves of the privileges of membership of our Association. The result was as disappointing as it possibly could be. Beyond the professors and lecturers of our three University Colleges, I found only two or three names with Welsh addresses, and there was no indication whatever that the members in question were teachers of mathematics in any of the schools of the Principality. Now, Wales is a country whose educational system has developed with surprising rapidity within the last few decades, and I can only conclude that a similar lack of support would be found to exist in other districts if the list were studied by any member possessing some knowledge of the educational systems of the districts in question.

It is true that, taken as a whole, our membership roll figures up to a very respectable total, but the greater proportion of the members are mathematicians who are qualified to benefit our Association by the contribution of papers to our *Gazette*, and by the expression of their opinions on questions connected with the teaching of mathematics; and the large body of mathematical and science masters who would derive benefit from joining our ranks do not appreciate the advantages to be derived from membership.

Under the conditions which unfortunately prevail in Great Britain, it is impossible for the average student to reach anywhere near the forefront of knowledge in any single branch of mathematics during the brief period of his University curriculum. As soon as this course is finished he usually obtains a post in a school and is cut adrift from those influences which would be of the greatest value in enabling him to form a clear concept of the aims and objects of mathematical teaching. He is unable to read the current literature on any branch of mathematics, and the only connection between him and the outside world is effected through syllabuses, text-books and examinations. He thus gets to regard mathematics more and more as a mere machine for the purpose of passing examinations. He may teach geometry on modern lines, and his pupils, instead of finding the greatest common measure of long quantics (a process

of which the only application I know of is in connection with Sturm's Theorem, and is unknown to most teachers), may be drilled in plotting graphs on squared paper and drawing triangles and measuring their areas; but in any case the pupils must in the long run regard mathematics only as an examination subject.

It may be urged that mathematics so taught is better than no mathematics at all, and that the great majority of schoolboys derive much benefit from being put through the examination mill, as part of a general education. But scholarships and bursaries have been founded in large numbers for the express purpose of finding out the genius wherever he exists and enabling him to work his way to the front, and a great deal of mischief may be done if the genius in question is not allowed proper scope for developing any ideas beyond the syllabus of the Leaving Examination.

It is a matter of common experience that the boys who win top scholarships on entrance to their University courses, whether at the older or at modern Universities, often turn out much less successful than those who have only done moderately well in their entrance examinations. In many cases such boys have been well primed up to a certain point and can answer even difficult questions up to that point, but their training has unfitted them for assimilating the new ideas which are required to carry them beyond that point.

This must necessarily occur unless the teacher is capable of contemplating a far wider vista of knowledge than that which he actually teaches.

A teacher of music may be engaged exclusively in teaching scales and five-finger exercises, and may be incapable of rendering a difficult sonata or concerto, but such a teacher is generally in a position to hear the works of the great masters performed by the leading artistes of the day, and without such outside influence the teaching would degenerate into mere useless drudgery. What we want is to place our mathematical teachers in the same position. That our present system of training teachers for schools, whether secondary or elementary, is defective, must I believe be very generally admitted, and this applies quite as much to the teaching at the older as at the newer Universities; but it is no part of my present purpose to attack this very difficult question. But seeing what the position now is, what is the most promising means of bringing the teacher in the provincial or small village school into contact with some of the mathematical work that is being done by men of ideas at the present time?

This question has been gone into pretty fully both by myself and Mr. Hilton, and we have both arrived at the conclusion that

the *Mathematical Gazette* is the one mathematical journal best suited to the requirements of the average graduate in arts or science who has taken the usual mathematical curriculum.

What then are the causes which deter the ordinary or the small mathematical master from joining our ranks? These causes might in most cases come under the following list:

1. He has never had the Association brought before his notice in such a way as to make him think of joining it.

2. He has not regarded membership of the Association as suitable for "the likes of him."

3. He has been frightened off by the somewhat forbidding appearance of certain papers published in the *Gazette*.

4. In the present heavily-taxed state of our country, the few extra shillings required for the annual subscription are an important consideration.

Now the last-named cause has undoubtedly laid low many scientific societies and journals that were in a very flourishing state in the good old times of a quarter of a century ago. It will have to be reckoned with as an obstacle to present progress, but I do not personally believe that a reduction of the annual subscription would make much difference. At any rate, the best way of bringing about such a reduction, if desirable, would be by a large increase in our membership roll. It must further be remembered that the class of members which we wish to enrol is not necessarily drawn from that particular class of society which has been most hard hit by the existing economic conditions of the country.

In regard to the other causes, the obvious and the best remedy is—advertising. A great deal can undoubtedly be done by calling the attention of students to the advantages of joining the Association when they go down from college, and this can well be done by tutors and college lecturers. But if the movement is to be successful, a circular should be sent both to the colleges for this purpose and to the mathematical masters in the schools, pointing out the desirability of bringing the Association into closer contact with all grades of mathematical teaching, and that it is to the interests of all grades of teachers to join our ranks, both for their own sake and in furtherance of this object.

No mathematical teacher in even the most elementary school should regard himself as disqualified from joining our ranks by any shortcomings in his own mathematical training. Such shortcomings may disqualify him from being a competent teacher, but this should be all the more reason for his bringing himself into closer touch with our organisation.

The Circolo matematico di Palermo has issued a circular in Italian, English, French and German, which has been sent to mathematicians all over the world, and has resulted in a very

large increase* in the membership roll of this now international mathematical society. A similar invitation issued on behalf of the French Physical Society to English physicists has resulted in a large number of workers in this country joining hands with their French colleagues.

If such a plan as I propose can be effectively carried out, Wales alone ought to contribute a hundred to our list, and this list would in a short time assume the nature of a Directory of Mathematical Teachers.

The present is an excellent time for making the effort, in view of the important part that Committees of our Association have played in bringing about the object originally contemplated in our foundation, namely, the Improvement of Geometrical Teaching. From the conditions existing in this country, recommendations or suggestions of committees, if effect be given to them, necessarily assume the form of legislative measures in their application to a large body of teachers. It is not my intention at present to discuss the desirability or otherwise of this position, but so long as it exists, it is important that the body responsible for framing or adopting recommendations should be representative, and should include all those who are affected by the proposed changes. The principles of constitutional government should be aimed at in connection with the teaching profession as a first step towards freedom of teaching.

The interests of English mathematicians have been greatly hampered in the past by the absence of any strong influential society comparable with the German Mathematiker Vereinigung and the American Mathematical Society.

The German Society has given abundant proof of its activity in the publication of the *Mathematical Encyclopaedia*. Further proofs were not wanting at the Mathematical Congress in 1904. The seven representatives of Great Britain who attended the Heidelberg meeting found themselves in the midst of a powerful body of men, full of energy and enterprise, standing high in popular estimation and strongly backed in their efforts by the support which they enlisted from municipal and State sources. The disabilities under which they laboured at home were well known to their hosts.

Take again the American Society, with its numerous local branches, its summer meetings and *colloquia*, and last, but not least, its *Bulletin*. This *Bulletin* might not inappropriately be described as the mathematical newspaper of the whole civilised

	1900	'01	'02	'03	'04	'05	'06
*New members admitted, -	6	18	11	14	21	110	84
Total membership, - -	180			188	195	255	**403**

(The 84 in '06 is from Jan.—July only.)

world. The courses of lectures at Halle are enumerated side by side with those at Harvard, and the papers read at the mathematical sections of the Deutsche Naturforscher Versammlung are chronicled no less thoroughly than those contributed to the American Association for the Advancement of Science. The award of the Smith's prizes, the papers set in the Cambridge Tripos, obituary notices, and appointments to chairs and lectureships, whether at Belgrade, Berlin, Bologna or Baltimore, all receive notice.

We here stand in the greater need of some equally powerful organisation, in view of the prevailing popular prejudice—I might almost describe it as insular prejudice—that exists against matters mathematical. Such a body must make a serious effort to win popularity, and must not merely appeal to the limited class of individuals that are engaged in higher mathematical research. The Mathematical Association is eminently fitted to step in and fill the gap. The changes in the teaching of geometry are just making themselves universally felt in the country, and this circumstance alone gives us a good start towards securing the confidence of the public.

Granting then an increase in our membership, in what direction should we extend the usefulness of our work? This can be done in more ways than one:

1. The encouragement of the use of the Queries column of our *Gazette* for the solution of difficulties however trivial. This will give the local members the feeling that wherever they may be placed they can rely on the help of the Association in case of need. A further result will be the more ready detection of fallacies in text-books, leading to the substitution of improved methods.

2. The foundation of local branches, affording facilities for intercourse between mathematical teachers residing near one another. This will naturally follow upon an increase in the membership roll.

3. The publication of more current news and personal information, including notices of appointments in our *Gazette*. In particular, there is no reason why we should not secure advertisements of mathematical scholastic vacancies.

4. *The holding of colloquia or courses of lectures on higher mathematical methods, delivered to teachers.* Such colloquia have been a great success in the United States. They are badly wanted in this country, and would do more than anything else to improve the status of our existing teachers.

5. The publication of lists of books suitable for teachers to read. By this I mean books giving a general survey of the scope and character of modern mathematical methods without assuming more advanced previous knowledge than is possessed by the

average teacher. The recent English translations of Dr. Mach's *Space and Geometry*, or Poincaré's *Science and Hypothesis* would be suitable for the purpose. Books on the history or philosophy of mathematics would of course figure largely in the list, and De Morgan's works might specially be cited.

6. Suggestions towards the opening up of avenues of easy approach to the study of higher mathematics. No better example of this want can be found at the present time than the Theory of Groups. The demand for an easy introductory treatise in this case will probably soon be filled by Mr. Hilton's forthcoming work, but a similar want is felt in nearly every branch of mathematics. I believe the want could be in many cases filled by a series of articles giving outlines of main principles without unnecessary technical details.

7. Finally, it is urgently important that our Editor should receive a more adequate remuneration for the services which he renders in connection with the *Gazette* than is possible with our present membership.

There has been a general wish on the part of many mathematicians that the Mathematical Congress should some day meet in Great Britain, but difficulties arose at the last Congress owing to the lack of organisation of our mathematicians, and difficulties of this character could be obviated if an influential body of the kind here suggested were to exist. I can only hope that the Mathematical Association will step into the stronghold that is awaiting its occupation, and will thus add to the valuable services which it has rendered in the past. G. H. Bryan.

THE TEACHING OF GEOMETRY.*

In the teaching of mathematics as of other subjects the primary determining factor ought to be, but seldom has been, the consideration of the pupil himself—his interests, aptitudes, and capabilities. The teacher should be free to devise the best means of securing his fullest co-operation, and examinations should allow the greatest possible elasticity. It is the reversal of the rational order that has wrought so much harm in the past and that so long delayed and still impedes reform.

Although an excellent start has been made, a little observation and independent thought soon convinces us that much more is yet possible in the way of reform with respect both to subject-matter and to the method of its presentation. For, even in spite of his reforming tendencies, a teacher is often in the clutch of the old system; his previous training and habits of thought make it difficult for him to look afresh at the subject from the tyro's

* Summary of lectures given at the College of Preceptors, Nov. and Dec., 1906.

standpoint, and he finds himself imposing his knowledge, his expressions, and mode of thought on his pupil. Instead of being frankly pre-Euclidean he enters half-apologetically on an experimental course, and hurries him on with haste to drink of "the pure milk of the word." By consciously basing his work on axioms, definitions, and the like he directs his pupil backwards instead of forwards—a course fatal to fresh thinking and out of harmony with the methods of modern inquiry. In his practice he fails to recognise that a mainly deductive course is highly unsuitable and quite the reverse of stimulating to the untrained mind.

On the other hand, a teacher avowedly attached to the old order would find it enriched and revitalised by aid of modern conceptions. Thus definitions get a reality they so frequently lack; an insignificant proposition such as Euc. I. 43 gains immensely in value and suggestiveness; while in such a one as II. 11, instead of everything being told out in usual fashion, an excellent method of approach lies to hand—

(1) A rough estimate as to the position of the point of medial section.

(2) The point fixed more definitely by a graph.

(3) Its exact position determined by Algebra, that is, by a modern method which not only gives both the "Practical Geometry" construction and that of Euclid, but also the negative solution.

Apart from the complete generalisation of the original problem, the superiority of the treatment suggested by the more modern developments must be apparent to anyone. Moreover, each of the three methods has its own special value.

It is well perhaps to point out to opponents of the new system that not all the errors of a transition stage are necessarily due to, or inseparable from, reform methods. Thus one might have anticipated a surfeit of practical work without definite purpose beyond the mere doing; a want of discretion as to whether a figure shall be a freehand sketch or shall be drawn to scale, in other words, a want of adaptability to new conditions and failure to perceive what is essential or desirable; impotence in drawing proper conclusions from data that are approximate, a power so valuable and necessary in everyday experience; and a want of critical faculty generally in dealing with new problems. Moreover, whatever laxity of reasoning is discoverable in the present-day pupil, it will not be overlooked that it was in the old school that his teacher was trained.

But, quite apart from any systems, our obvious course is to remedy what defects we can and to avail ourselves of the best both of past and present times. Thus geometry has much to gain by being brought into closer relation with other subjects

and into completer touch with life. Formulae and technical terms and statements may be, and to beginners often are quite uninspiring; translate them into terms of one's experience and they receive actuality. Introduce the consideration of congruent triangles by reference to the triangulation of a country, have a class discussion on lattice-work in treating of quadrilaterals; illustrate, where possible, by the work of the builder or the engineer. Thus the circle in mechanism has, for a boy, a charm and reality that the traditional circle of geometry does not possess, and a belt passing over wheels calls forth a response that the mention of common tangents cannot evoke. Relate the subject to woodwork, drawing and design, geography, optics; let arithmetic, algebra, and geometry cease to exist as such, and become mathematics—in actual school work, and in examinations generally. There is much work yet possible for reformers in the formation of attractive and more profitable subject-matter to replace the stale and stodgy exercises that have long since done more than their duty. This reform is second in importance only to the much-needed one for the right presentation of the subject, the overthrow of the 'tell everything' method and the fostering of the spirit of investigation. By thus enlisting the interest of the pupil and bringing into play his observation, imagination, and theorising, a true scientific method is possible. The young inquirer submits his guesses to tests, and proves or disproves by whatever methods he can use. It follows, as a matter of course, that we cannot dissever practice and theory; they are interlinked, inseparable as induction and deduction in all scientific investigations. We make use of apparatus, mathematical instruments and also of different branches of mathematics just as occasion demands, that is, not merely for illustration and verification, but for all the various stages in an inquiry.

With regard to a preliminary course, one finds that the idea that young children can do things but cannot reason is responsible for much bad and dull teaching. For perpetual drawing and measuring without objective can be as dreary as you please, especially if there is much needless instruction how to do everything and little scope for a lad's self-expression and the satisfying of his natural curiosity and love of discovery. And yet all the while this experimental and practical work could be made wonderfully stimulative of thought and inquiry, and this implies simple talks and discussion. But the beginner must be supplied with proper material for reasoning upon, and we must start not from our own, but from his point of view—his wrong guesses and faulty reasoning are often most instructive. In this course, too, the reasoning will not be of the most general kind, except occasionally. The procedure indeed will

often reverse that of the old order, and will not continue the absurdity, from a teaching point of view, of starting off from the widest of generalisations, as usual with the Index Laws for instance.

Further, a lad likes the sense of forging ahead—he must cover ground. The preliminary course must not be a snippet from a more extended one, but should have a completeness and character of its own, dealing with the salient points in Solid as well as in Plane Geometry in an elementary way, but not superficially. In the treatment of solid geometry there need be no lack of variety. By using any plastic material the various solid forms are readily represented, their volumes determined, and any section at once shown. Jars and bottles of different shapes are also useful for this purpose, and woodwork in particular is very suggestive. Time should not be wasted on elaborate or highly finished models; a half model obtained by folding paper is often more instructive than a complete one. Pupils should work out the designs for cardboard models themselves, imagining one cut along the edges till the surfaces are all brought into one plane, and, when formed, they should be put to practical use; for instance, for determining approximately the relation between the volumes of pyramid and prism with the same base and height. Then, again, the generation of a solid, as suggested by a lift or a spinning top, is instructive as well as entertaining. Models of cardboard and string, wire models, and hat-pins stuck through cardboard, have their special uses, the latter in connection with the locating of a point in space for example. In considering the sphere, there is the obvious connection between geography and geometry, to the mutual advantage of each. The mathematical teacher is in danger of overlooking homely illustrations; he thinks of the geometrical sphere, and the boy of the ball for cricket or football. Formulae get much abused at times; that for the most irregular of wedges, for instance, is foisted on to a raw schoolboy, and to his great disadvantage. On the other hand, the drawing of a simple solid and the interpretation of its conventional representation gets little attention, as shown by the incompetence of youths even of 18 or more.

Simple area work—not by rule of thumb or formulae—and the idea of Ratio should come quite early in a preliminary course, and should recur throughout. Also, and very important, the pupil should be familiarised with quantities that do not vary proportionally. The admirable examples provided by geometry, and forcefully represented by graph work, could do much towards making the arithmetical treatment of proportion far more intelligent than is customary. The right feeling, too, for approximation can best be given through geometry—again

not by rules, which are too often applied in mechanical fashion and the significance of the process lost. In this connection, too, we shall deal with areas and volumes of figures with curved and irregular boundaries, figures which are always met with in real life, but scarcely ever in a geometrical course.

Again, a beginner—because he is a beginner—finds a great difference between the consideration of a special and of the more general case, a fact that has often been strangely overlooked in mathematics. By fully recognising this—in the treatment of angles, for instance—far greater advance would be possible, and the pupil would be rationally led up to the generalisation. Neither in defining nor in generalising did the old system give any training, since definitions and the like were always supplied ready made at the outset.

Probably the most effective way of dealing with congruent triangles is from the constructional point of view. Quite early, triangles will have been copied or drawn from specified data, and examples introduced containing superfluous or inadequate data. In a subsequent class discussion as to what is necessary and sufficient to determine a triangle, a piece of apparatus formed of two jointed rods for sides, and a variable length of string for base is very useful; a great advantage of such a figure is that connected truths are brought together for consideration. Here, too, and in connection with parallels, much might be suggested with regard to similar figures and the relation between corresponding lines and areas, and some idea given of the value of radial projection. Considering the far-reaching importance of this, and that it can be dealt with in a delightfully simple and interesting way, its postponement to what for many is the never never time needs some justification. Although the method of superposition has a practical value for testing the accuracy of figures drawn to specified data, one cannot but feel a sense of unreality in a lad drawing (or imagining) two triangles with sides 4 in. and 3 in. and included angle of 30°, say, and then solemnly proceeding to suppose one placed on the other to determine whether they must of necessity fit. If two circles have each a radius of 3 in., are we to prove their equality in the same way?

Variety in treatment of the same geometrical truth (*e.g.* Euc. I. 47) is an advantage, since different presentations appeal to different minds, and in the case of experimental work there is the cumulative value of evidence as well as the greater suggestiveness.

The geometry of the circle lends itself admirably to experimental treatment, and here, as in the case of work with areas and volumes, the value of figures admitting of change of form is evident; the sense of continuity can be so well given by their aid, and so can telling illustrations of loci, and, along with

graph work, the change in a variable quantity can be so well shown. In the case of a chord moving towards the centre of a circle, the *fact* of change is obvious to a boy, the interest is in the *nature* and *rate* of the change, and it is at this, the most effective time, that the change in the sine, cosine, and tangent of an angle should be placed on record by a graph, and this put to practical use in the solution of a right-angled triangle. This, moreover, is possible without even the necessity of premature mention of these technical terms.

A preliminary course similar in extent and treatment to that roughly outlined above seems the most fitting introduction to the subject. If a boy worked through this—and any ordinary boy and girl easily can—and went no further, he would have done something worth doing in itself; if he proceeds further, it forms an admirable foundation to build upon, and rapid progress would be possible; indeed, when the chief points have been thus reconnoitred, the so-called 'logical' treatment is comparatively simple. Here, again, far more free class discussion is needed, if only for bringing out defects in reasoning which otherwise escape detection. Indeed, the much exaggerated sequence difficulty may prove an advantage, for how is one to really learn what reasoning in a circle means, if he is always to go along a path marked out for him by authority; or how learn the danger of false analogy as between plane and solid geometry for example —an error more common perhaps than is usually suspected; or the advantage or disadvantage of considering special cases? In short, one cannot learn to discuss without discussing, of which we have next to none in our school or university systems; he cannot learn to judge the value of evidence or proof if he is accustomed to one kind only, and that, one which excludes real cases, and keeps strictly to the hypothetical; nor can he learn to reason as a modern if his models and practice are just those of the ancients, Euclid, Aristotle or others. The fatal objection against the old system may still lie against the new if after some shuffling of propositions and shedding of extraneous matter we follow on much in the same track as before instead of proceeding along the lines so clearly marked out by scientific investigators.

J. G. HAMILTON.

REVIEW.

A System of Applied Optics. By H. DENNIS TAYLOR. (Macmillan.)

This is a book on optics by a practical optician. It is almost, if not quite, the first book of its kind written by an Englishman, and as such, as well as for its intrinsic merits, it deserves a warm welcome. In Germany the modern literature of optics is in closest alliance with the great optical industry of the country—an alliance of theory and

practice which explains the notable success of the latter. But in this country our optical books have been written with a view to mathematical examinations, and are beautifully remote from the pieces of glass which we use in our cameras and microscopes. The lenses, in fact, serve as an excuse for the mathematics. With regard to Mr. Taylor's book also we may say, in a different sense, that the lenses are an excuse for the mathematics. His formulae are at times terrific, and his analysis seems, to the merely academic mind, somewhat lacking in elegance, but these things do not count against the satisfaction the reader feels in the real connexion between the results obtained and the actual production of good lenses. Indeed, the author tells us how his theoretical investigations guided him towards the design of the well-known "Cooke" forms of lens-combinations.

In the preface reference is made to the important advances in optical theory due to Abbe and his school, and to Seidel, but the author does not adopt any of their methods. He bases his work on that of Coddington, whose formulae he extends, in the first instance, by analysis of wide instead of narrow pencils. At an early stage in the book the method of "elements" is introduced, and is a prominent feature of all that follows. This consists in the device of regarding a thick lens as made up of two thin lenses at the surfaces, each with a flat face on the inner side, and separated by a parallel plane glass plate. After preliminary sections on first order formulae and the equivalent focal lengths of systems of thin lenses, the corrections for width of pencil, obliquity, and eccentricity of incidence are discussed in succession. These corrections are finally embodied in two lengthy formulae, applying to the primary and secondary planes, and giving the points of convergence of extreme rays of a pencil falling obliquely on a lens through a stop of given aperture and position. The terms are clearly grouped according to their dependence on thickness, spherical aberration, normal curvature error, eccentricity correction associated with spherical aberration, and eccentricity correction associated with coma. In following sections we have an admirable discussion of the bearings of the analytical results on the actual problem of lens design, then a detailed account of coma, with reproduction of some of Prof. Thompson's interesting photographs; and finally, sections on distortion and achromatism. Through the whole book the author's practical acquaintance with the matters under discussion is constantly in evidence.

It is natural to compare the book under review with the recently published book, *Die Bilderzeugung in optischen Instrumenten*, produced in collaboration by the scientific workers of the firm of Zeiss in Jena. One misses the elegance of method and the generality of treatment which are such conspicuous features of the German book. But against these we must set the absolute straightforwardness of Mr. Taylor's mode of attack, the thoroughness of his discussion of the special formulae which he evolves, and the simplicity of his mathematical methods. By these qualities he will appeal to many readers who would fail to gather very definite ideas from the other book mentioned.

W. B. MORTON.

QUERIES.

The recent Cambridge Senior Local Examination Paper on the Differential Calculus.

I should be very glad to read the opinions of members of the M.A. regarding the above paper.

It seems to me that the teaching of the elements of the Differential and Integral Calculus to classes of *average power* is very desirable and might be encouraged by Examining Boards.

My own opinion, however, is that such examination papers should consist almost entirely of numerical questions.

Take for example Question 9:

"Find the equation of the tangent to the curve $y=f(x)$ at the point (x, y) and write down the form it takes when x and y are given in terms of a variable parameter."

I should prefer to see some such question as

"Find the equation of the tangent at the point where $x=\frac{\pi}{12}$ to the curve whose equation is $y=3\sin 2x$.

"At what points will the tangents be parallel to the axis of x?," etc.

A. F. VAN DER HEYDEN.

NOTICE.

It is proposed to start an International Mathematical Journal written in Esperanto, costing about 10s. per annum. For further information apply to F. J. VAES, Mathenesserlaan 290, Rotterdam (Holland). Letters, $2\frac{1}{2}$d.; postcards, 1d.

BOOKS, ETC., RECEIVED.

Algebraical Examples. By W. G. BORCHARDT. Pp. viii, 492, lxi. 3s. 1906. (Rivington.)

The Manufacture of Light. By SILVANUS P. THOMPSON. Pp. 67. 1s. net. 1906. (Macmillan.)

Wiadomości Matematyczne. Edited by S. DICKSTEIN. Tom. x. Zeszyt 4-5-6. 1906. (Sikorskiego, Warszawa.)

The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXVIII. No. 4. Oct. 1906.

Functions of Three Real Independent Variables. By H. L. COAR. *An Invariant Condition for certain Automorphic Algebraic Forms.* A. B. COBLE. *On some Cases of Motion of a Solid in Infinite Liquid.* By G. Kolosoff. *On the Arrangement of the Real Branches of Plane Algebraic Curves.* By V. RAGSDALE.

Annaes Scientificos da Academia Polytechnica do Porto. Edited by F. G. TEXEIRA. Vol. I. No. 4. 1906. (University Press, Coimbra.)

Fourier's Series and Integrals. By H. S. CARSLAW. Pp. xvii, 434. 14s. net. 1906. (Macmillan.)

Theorie des Integrallogarithmus und Verandter Transzendenten. By NIELS NIELSEN. Pp. vi, 106. 3 m. 60. 1907. (Teubner.)

Gleichgewichte der Stereomeren. By W. MEYERHOFER. Pp. 71. 2 m. 40. 1907. (Teubner.)

Vorlesungen über Differential- und Integral-Rechnung. II. By E. CZUBER, Pp. viii, 532. 12 m. 1907. (Teubner.)

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.